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Multi-particle correlations in qp-Bose gas model

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Abstract
The approach based on a multimode system of q-deformed oscillators and the related picture of an ideal gas of q-bosons enables us to effectively describe the observed non-Bose-type behaviour, in experiments on heavy-ion collisions, of the intercept (or the 'strength') λ of the two-particle correlation function of identical pions or kaons. In this paper we extend the main results of that approach in two aspects: first, we derive in explicit form the intercepts of n-particle correlation functions in the case of the q-Bose gas model and, second, provide their explicit two-parameter (or qp-) generalization.

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1. Introduction

Quantum and q-deformed algebras are known to be very useful in diverse problems in many branches of mathematical physics and modern field theory [1, 2], as well as in molecular/nuclear spectroscopy [3]. Equally fruitful should be their direct application in the phenomenology of particle properties (see [4–7], and also [8] with references therein). Recently, it has been demonstrated [9] that the use of multimode q-deformed oscillator algebras along with the related picture of an ideal gas of q-bosons (q-Bose gas model) proves its efficiency in modelling the unusual properties of the intercept λ of the two-particle correlation function, which is the measured value corresponding to zero relative momentum of two identical mesons, pions or kaons, produced and registered in relativistic heavy-ion collisions [10], where λ exhibits a sizable observed deviation from the naively expected purely Bose–Einstein-type behaviour. The model predicts [11, 12], for a fixed value of q, the exact shape of dependence of the intercept $\lambda = \lambda(\mathbf{K})$ on the pair mean momentum $\mathbf{K}$ and suggests asymptotic coincidence of λ_π and λ_K of pions and kaons. Put in other words, the intercept λ, being connected directly and unambiguously with the deformation parameter q, tends in the limit of large pair mean momentum to a constant, less than unity, determined just by q and shared by pions and kaons. It is worth noting that confronting the predicted λ_π behaviour with the corresponding data from STAR/RHIC shows good agreement [12], at least in the case of two-pion correlations.

While two-particle correlations are known to carry information about the spacetime structure and dynamics of the emitting source [10], in connection with some recent experiments it was pointed out [13, 14] that taking into consideration, in addition to the single-particle spectra and two-particle correlations based analysis, the number of data concerning three-particle correlations provides important supplementary information on the properties of the emitting region, valuable for confronting theoretical models with concrete experimental data. Likewise, study of four- and five-particle correlations is also desirable [15]. All that motivates the main goal of this paper is to derive the explicit formulae for the intercepts of higher order (n-particle, with $n \geqslant 3$) HBT correlations. Moreover, below we will obtain in explicit form the intercepts $\lambda^{(n)}$ of n-particle correlations for the extended version of the developed approach when one uses the two-parameter qp-deformation of bosonic oscillators and the model of a gas of qp-bosons.

Right from the very beginning, and up to the present, in the applications of the q-algebras to the phenomenology of hadrons there is growing evidence [6, 8] that the phase form of the q-parameter is of great importance. Therefore, we hope that possessing the most general formulae for n-particle correlations and confronting them with the data from contemporary experiments will be helpful in clearing up the actual preference of choosing the form $q = \exp(\mathrm{i}\theta)$ of the deformation parameter. It is just this alternative for the choice of the deformation parameter q that implies a very attractive physical interpretation of the q-parameter as the one that is directly linked to the mixing issue of elementary particles, either of bosons [6, 16] or fermions [8, 17].

The paper is organized as follows: section 1 contains a sketch of necessary preliminaries concerning the two most popular types of q-deformed oscillators, as well as their two-parameter or qp-generalization. In section 2 we discuss basic points of the approach based on the q-Bose gas model, along with consideration of single-particle q-distributions. The remaining two sections are devoted to the properties of two- and three-particle correlation functions, and to the main topic of this paper—the results on the multi-particle (nth order) correlations, for the algebras of both the q-deformed and the qp-deformed versions of generalized oscillators. Details of the derivation of basic formulae are relegated to the appendix.

2. *q*-deformed and *qp*-deformed oscillators

We begin with a necessary set-up concerning two types of q-deformed oscillators, and also their two-parameter generalization.

2.1. *q*-oscillators of AC type

The q-oscillators of the Arik–Coon (or AC) type are defined by the relations [18, 19]

$$\begin{aligned} &a_i a_j^{\dagger} - q^{\delta_{ij}} a_j^{\dagger} a_i = \delta_{ij} \qquad [a_i, a_j] = \left[a_i^{\dagger}, a_j^{\dagger}\right] = 0 \\ &[\mathcal{N}_i, a_j] = -\delta_{ij} a_j \qquad \left[\mathcal{N}_i, a_j^{\dagger}\right] = \delta_{ij} a_j^{\dagger} \qquad [\mathcal{N}_i, \mathcal{N}_j] = 0 \end{aligned} \tag{1}$$

where $-1 \leqslant q \leqslant 1$. Note that this is the *system of independent* q-oscillators as clearly seen at $i \neq j$.

From the vacuum state given by $a_i|0, 0, \ldots\rangle = 0$ for all i, the basis state vectors

$$|n_1, \ldots, n_i, \ldots\rangle \equiv \frac{1}{\sqrt{\lfloor n_1\rfloor!\lfloor n_2\rfloor!\cdots\lfloor n_i\rfloor!\cdots}} \left(a_1^{\dagger}\right)^{n_1}\left(a_2^{\dagger}\right)^{n_2}\cdots\left(a_i^{\dagger}\right)^{n_i}\cdots|0, 0, \ldots\rangle \tag{2}$$

are constructed as usual, so that

$$a_i^\dagger|\ldots,n_i,\ldots\rangle=\sqrt{\lfloor n_i+1\rfloor}|\ldots,n_i+1,\ldots\rangle \qquad a_i|\ldots,n_i,\ldots\rangle=\sqrt{\lfloor n_i\rfloor}|\ldots,n_i-1,\ldots\rangle$$
$$\mathcal{N}_i|n_1,\ldots,n_i,\ldots\rangle=n_i|n_1,\ldots,n_i,\ldots\rangle. \tag{3}$$

Here the notation $\lfloor\ldots\rfloor$ for so-called basic numbers and the corresponding extension of factorial, namely

$$\lfloor r\rfloor\equiv\frac{1-q^r}{1-q} \qquad \lfloor r\rfloor!\equiv\lfloor 1\rfloor\lfloor 2\rfloor\cdots\lfloor r-1\rfloor\lfloor r\rfloor \qquad \lfloor 0\rfloor!=\lfloor 1\rfloor!=1. \tag{4}$$

are used. The q-bracket $\lfloor A\rfloor$ for an operator A is understood as a formal series. At $q\to 1$, from $\lfloor r\rfloor$ and $\lfloor A\rfloor$ one recovers r and A, thus going back to the formulae for the standard bosonic oscillator. For the *deformation parameter* q such that $-1\leqslant q\leqslant 1$, the operators $a_i^\dagger, a_i$ are conjugates of each other.

For $q\neq 1$, the bilinear $a_i^\dagger a_i$ depends nonlinearly on the number operator $\mathcal{N}_i$:

$$a_i^\dagger a_i=\lfloor\mathcal{N}_i\rfloor \tag{5}$$

so that at $q=1$ the familiar equality $a_i^\dagger a_i=\mathcal{N}_i$ is recovered.

2.2. *q-oscillators of BM type*

The q-oscillators of Biedenharn–Macfarlane (BM) type are defined by the relations [19, 20]

$$\begin{array}{lll}[b_i,b_j]=\left[b_i^\dagger,b_j^\dagger\right]=0 & [N_i,b_j]=-\delta_{ij}b_j & \left[N_i,b_j^\dagger\right]=\delta_{ij}b_j^\dagger \\ [N_i,N_j]=0 & b_ib_j^\dagger-q^{\delta_{ij}}b_j^\dagger b_i=\delta_{ij}q^{-N_j} & b_ib_j^\dagger-q^{-\delta_{ij}}b_j^\dagger b_i=\delta_{ij}q^{N_j}.\end{array} \tag{6}$$

In this case the extended Fock space of basis state vectors is constructed in a way similar to the above case, with the only modification that now we use, instead of basic numbers, the q-bracket and q-numbers, namely

$$b_i^\dagger b_i=[N_i]_q \qquad [r]_q\equiv\frac{q^r-q^{-r}}{q-q^{-1}}. \tag{7}$$

Formulae similar to (2)–(5) are valid for the operators $b_i, b_j^\dagger$ if, instead of (4), we now use definition (7) for the q-bracket. Clearly, the equality $b_i^\dagger b_i=N_i$ holds only in the 'no-deformation' limit of $q=1$. For consistency of the conjugation, we put

$$q=\exp(\mathrm{i}\theta) \qquad 0\leqslant\theta<\pi. \tag{8}$$

2.3. *qp-oscillators*

Besides the q-bosons of AC-type and BM-type, in what follows we will also consider the two-parameter (or qp-) generalization of deformed oscillators given by the relations [21]

$$\begin{array}{ll}[N^{(qp)},A]=-A & [N^{(qp)},A^\dagger]=A^\dagger \\ AA^\dagger-qA^\dagger A=p^N & AA^\dagger-pA^\dagger A=q^N\end{array} \tag{9}$$

from which

$$A^\dagger A=[\![N^{(qp)}]\!]_{qp} \qquad \text{with} \qquad [\![X]\!]_{qp}\equiv\frac{q^X-p^X}{q-p}. \tag{10}$$

In this definition we have shown only one mode, although in what follows we will deal with the system, like in (1) or (6), of independent (that is, mutually commuting) copies/modes of the qp-deformed oscillator. Note that X in (10) can be either a number or an operator. Clearly, putting $p=1$ immediately leads us to the AC-case while putting $p=q^{-1}$ reduces it to the BM-type of q-bosons.

3. Statistical *q*-distributions

For the dynamical multi-particle (say, multi-pion or multi-kaon) system, we consider the model of an ideal gas of q-bosons (IQBG) by taking the free, or non-interacting, Hamiltonian in the form [22–24]

$$H = \sum_i \omega_i \mathcal{N}_i \tag{11}$$

where $\omega_i = \sqrt{m^2 + \mathbf{k}_i^2}$, $\mathcal{N}_i$ is the number operator given in (5) or (7) or (10) and the subscript i labels different modes. Let us note that among a variety of possible choices of Hamiltonians, choice (11) is the unique non-interacting one, which possesses an additive spectrum (see [22, 23]). Clearly, it is assumed that the 3-momenta of particles take their values from a discrete set (i.e. the system is contained in a large finite box of volume $\sim L^3$).

To obtain basic statistical properties, one evaluates thermal averages

$$\langle A \rangle = \frac{\mathrm{Sp}(A\rho)}{\mathrm{Sp}(\rho)} \qquad \rho = \mathrm{e}^{-\beta H}$$

where $\beta = 1/T$ and the Boltzmann constant is set equal to 1. Calculating, say, in the case of AC-type q-bosons the thermal average $\langle q^{\mathcal{N}_i} \rangle$, with $\mathcal{N}_i$ from (5), with respect to the chosen Hamiltonian (11), we obtain

$$\langle q^{\mathcal{N}_i} \rangle = \frac{\mathrm{e}^{\beta\omega_i} - 1}{\mathrm{e}^{\beta\omega_i} - q} \tag{12}$$

and the distribution function (recall that $-1 \leqslant q \leqslant 1$) is found as [22, 23]

$$\left\langle a_i^\dagger a_i \right\rangle = \frac{1}{\mathrm{e}^{\beta\omega_i} - q}. \tag{13}$$

The usual Bose–Einstein distribution corresponds to the no-deformation limit of $q \to 1$. In the particular cases $q = -1$ or $q = 0$ the distribution function (13) yields respectively Fermi–Dirac or classical Boltzmann ones. Note that this coincidence is rather a formal one: the defining relations (1) at $q = -1$ or $q = 0$ differ from those for the system of fermions or the non-quantal (classical) system. The formal coincidence of equation (13) at $q = -1$ with the Fermi–Dirac distribution can be interpreted in terms of the impenetrability (or hard-core) property of such bosons. The difference with the system of genuine fermions lies in *commuting* (versus truly fermionic anticommuting) of *non-coinciding* modes at $q = -1$, see (1).

Now consider BM-type q-bosons. The Hamiltonian is chosen again as that of IQBG, but now with the number operator given in (7), i.e.

$$H = \sum_i \omega_i N_i. \tag{14}$$

Calculation of $\langle q^{\pm N_i} \rangle$ yields $\langle q^{\pm N_i} \rangle = (\mathrm{e}^{\beta\omega_i} - 1)(\mathrm{e}^{\beta\omega_i} - q^{\pm 1})^{-1}$. Then, from the formula $\left\langle b_i^\dagger b_i \right\rangle = (\mathrm{e}^{\beta\omega_i} - q)^{-1} \langle q^{-N_i} \rangle$ the expression for the q-deformed distribution function (note that $q + q^{-1} = [2]_q = 2\cos\theta$) follows (see also [22, 23]):

$$\left\langle b_i^\dagger b_i \right\rangle = \frac{\mathrm{e}^{\beta\omega_i} - 1}{\mathrm{e}^{2\beta\omega_i} - 2\cos\theta\, \mathrm{e}^{\beta\omega_i} + 1}. \tag{15}$$

Although the deformation parameter q is taken as *complex* according to (8), the explicit expression (15) for the q-distribution function shows that it is real.

It is easily seen that the shape of the function $f(\mathbf{k}) \equiv \langle b^\dagger b \rangle(\mathbf{k})$ in (15) is such that the q-deformed distribution function with $q \neq 1$ is intermediate relative to the other two curves, the standard Bose–Einstein distribution function and the classical Boltzmann one (the same is also evident for the above q-distribution function (13) of the AC-type q-bosons). That is, the deviation of the q-distribution (15) from the quantum Bose–Einstein distribution goes, when q goes away from the no-deformation limit $q = 1$, in the 'right direction', towards the classical Boltzmann one.

4. Two- and three-particle correlations of q-bosons

Although the formulae for two-particle correlation functions have been obtained earlier [9], we recall them here for the sake of a more complete exposition. In the remaining part of this section some new results will be presented. So, we consider two-particle correlations first with the AC-type of q-bosons. Starting with the identity

$$a_i^\dagger a_j^\dagger a_k a_l - q^{-\delta_{ik}-\delta_{il}} a_j^\dagger a_k a_l a_i^\dagger = \left[a_i^\dagger, a_j^\dagger\right] a_k a_l + a_j^\dagger \left[a_i^\dagger, a_k\right]_{q^{-\delta_{ik}}} a_l + q^{-\delta_{ik}} a_j^\dagger a_k \left[a_i^\dagger, a_l\right]_{q^{-\delta_{il}}}$$

where $[X, Y]_\kappa \equiv XY - \kappa YX$, by taking thermal averages of both sides we find

$$\langle a_i^\dagger a_j^\dagger a_k a_l\rangle = \frac{e^{\beta\omega_i} - q}{q^{1-\delta_{ik}-\delta_{il}} e^{\beta\omega_i} - q}\left(\langle a_j^\dagger a_l\rangle\langle a_i^\dagger a_k\rangle + q^{-\delta_{ij}}\langle a_j^\dagger a_k\rangle\langle a_i^\dagger a_l\rangle\right).$$

For coinciding modes this leads to the formula

$$\langle a_i^\dagger a_i^\dagger a_i a_i\rangle = \frac{1+q}{(e^{\beta\omega_i} - q)(e^{\beta\omega_i} - q^2)}. \tag{16}$$

From the last relation and the q-distribution (13) the ratio under question (called the intercept) gives the result:

$$\lambda_i \equiv \frac{\langle a_i^\dagger a_i^\dagger a_i a_i\rangle}{\langle a_i^\dagger a_i\rangle^2} - 1 = -1 + \frac{(1+q)(e^{\beta\omega_i} - q)}{e^{\beta\omega_i} - q^2} = q\,\frac{e^{\beta\omega_i} - 1}{e^{\beta\omega_i} - q^2}. \tag{17}$$

Note that in the non-deformed limit $q \to 1$ the value $\lambda_{\mathrm{BE}} = 1$, proper for Bose–Einstein statistics, is correctly reproduced from equation (17). This obviously corresponds to the Bose–Einstein distribution obtained in (13) at $q \to 1$. The quantity (intercept) λ is important because it can be directly confronted with empirical data. In this respect, let us note that there exists a direct asymptotic relation $\lambda = q$, which corresponds to the limit of large momentum or low temperature (in that case $\beta\omega \to \infty$).

We now go over to the Biedenharn–Macfarlane q-oscillators (6) and find the formula for the monomode two-particle correlations, i.e. for identical particles with coinciding momenta. From the relation

$$\langle b_i^\dagger b_i^\dagger b_i b_i\rangle - q^2\langle b_i^\dagger b_i b_i b_i^\dagger\rangle = -\langle b_i^\dagger b_i q^{N_i}\rangle(1+q^2)$$

valid for the monomode case at hand, we deduce

$$\langle b_i^\dagger b_i^\dagger b_i b_i\rangle = \frac{1+q^2}{q^2 e^{\beta\omega_i} - 1}\langle b_i^\dagger b_i q^{N_i}\rangle.$$

Evaluation of the thermal average on the rhs yields $\langle b_i^\dagger b_i q^{N_i}\rangle = q/(e^{\beta\omega_i} - q^2)$. Using this we find the expression for the two-particle distribution, namely

$$\langle b_i^\dagger b_i^\dagger b_i b_i\rangle = \frac{2\cos\theta}{e^{2\beta\omega_i} - 2\cos(2\theta)\, e^{\beta\omega_i} + 1}. \tag{18}$$

Then, the desired formula for the intercept of two-particle correlations of the BM-type q-bosons, with the notation $t_i \equiv \cosh(\beta\omega_i) - 1$, reads

$$\lambda_i = -1 + \frac{\langle b_i^\dagger b_i^\dagger b_i b_i\rangle}{\left(\langle b_i^\dagger b_i\rangle\right)^2} = \frac{2\cos\theta(t_i + 1 - \cos\theta)^2}{t_i^2 + 2(1-\cos^2\theta)t_i} \tag{19}$$

and again is a real function.

4.1. Three-particle correlations of the q-bosons of AC-type

Derivation of three-particle correlation functions proceeds analogously to the two-particle case. Considering the q-deformed oscillators of AC-type we start with the easily verifiable identity

$$a_j^\dagger a_k^\dagger a_l a_m a_s a_i^\dagger = a_j^\dagger a_k^\dagger a_l a_m \left[a_s, a_i^\dagger\right]_{q^{\delta_{is}}} + q^{\delta_{is}}\left\{a_j^\dagger a_k^\dagger a_l \left(\left[a_m, a_i^\dagger\right]_{q^{\delta_{im}}}\right) a_s + q^{\delta_{im}}\left(a_j^\dagger a_k^\dagger \left(\left[a_l, a_i^\dagger\right]_{q^{\delta_{il}}}\right) a_m a_s + q^{\delta_{il}} a_j^\dagger a_k^\dagger a_i^\dagger a_l a_m a_s\right)\right\}$$

and take thermal averages of both the sides. This leads to the equality

$$\left\langle a_i^\dagger a_j^\dagger a_k^\dagger a_l a_m a_s\right\rangle = \frac{\mathrm{e}^{\beta\omega_i} - q}{\mathrm{e}^{\beta\omega_i} - q^{\delta_{is}+\delta_{im}+\delta_{il}}}\left(\left\langle a_j^\dagger a_k^\dagger a_l a_m\right\rangle\left\langle a_i^\dagger a_s\right\rangle + q^{\delta_{is}}\left\langle a_j^\dagger a_k^\dagger a_l a_s\right\rangle\left\langle a_i^\dagger a_m\right\rangle + q^{\delta_{is}+\delta_{im}}\left\langle a_j^\dagger a_k^\dagger a_m a_s\right\rangle\left\langle a_i^\dagger a_l\right\rangle\right)$$

which in view of $\left\langle a_i^\dagger a_j\right\rangle = \delta_{ij}\left\langle a_i^\dagger a_i\right\rangle = \delta_{ij}/(\mathrm{e}^{\beta\omega_i} - q)$ (cf (13)), in the monomode $i = j = k = l = m = s$ case yields

$$\left\langle a_i^\dagger a_i^\dagger a_i^\dagger a_i a_i a_i\right\rangle = \frac{(1+q)(1+q+q^2)}{(\mathrm{e}^{\beta\omega_i} - q)(\mathrm{e}^{\beta\omega_i} - q^2)(\mathrm{e}^{\beta\omega_i} - q^3)}. \tag{20}$$

From the latter relation, dividing it by $\langle a_i^\dagger a_i\rangle^3$, we derive the desired expression for the intercept (or strength) $\lambda^{(3)}$ of the three-particle correlation function (we drop the label i)

$$\lambda_{\mathrm{AC}}^{(3)} \equiv \frac{\langle a^{\dagger 3} a^3\rangle}{\langle a^\dagger a\rangle^3} - 1 = \frac{(1+q)(1+q+q^2)(\mathrm{e}^{\beta\omega} - q)^2}{(\mathrm{e}^{\beta\omega} - q^2)(\mathrm{e}^{\beta\omega} - q^3)} - 1. \tag{21}$$

In a similar manner, it is possible to derive the (intercept of) three-particle correlation function for the system of BM-type q-bosons. However, intead of doing this, in the next section we will derive the most general results for both 3- and n-particle, $n > 3$, correlation functions in the two-parameter (i.e. qp-deformed) extension of bosons, from which the desired formulae for the BM-type of q-bosons will follow as particular cases.

5. *n*-particle correlations: *q*-bosons and *qp*-bosons

As an extension of equations (16), (20), it is not difficult to derive, using the method of induction, the following general result for the n-particle monomode distribution functions of AC-type q-Bose gas:

$$\left\langle\left(a_i^\dagger\right)^n (a_i)^n\right\rangle = \frac{\lfloor n\rfloor!}{\prod_{r=1}^n(\mathrm{e}^{\beta\omega_i} - q^r)} \qquad \lfloor m\rfloor \equiv \frac{1-q^m}{1-q} = 1 + q + q^2 + \cdots + q^{m-1}. \tag{22}$$

From this expression the desired formula for the intercepts $\lambda^{(n)} \equiv \frac{\langle a^{\dagger n} a^n\rangle}{\langle a^\dagger a\rangle^n} - 1$ of n-particle correlations of AC-type q-bosons immediately follows (with i dropped)

$$\lambda_{\mathrm{AC}}^{(n)} = -1 + \frac{\lfloor n\rfloor!(\mathrm{e}^{\beta\omega} - q)^{n-1}}{\prod_{r=2}^n(\mathrm{e}^{\beta\omega} - q^r)}. \tag{23}$$

In the asymptotics of $\beta\omega \to \infty$ (i.e. for very large momenta or, at fixed momentum, for very low temperature) the result depends only on the deformation parameter

$$\lambda_{\mathrm{AC}}^{(n)\mathrm{asympt}} = -1 + \lfloor n\rfloor! = -1 + \prod_{k=1}^n\left(\sum_{r=0}^k q^r\right) = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}) - 1. \tag{24}$$

This remarkable fact can serve as test one when confronting the developed approach with the numerical data for pions and kaons extracted from the experiments on relativistic heavy ion collisions.

Now we come to the base point.

5.1. Extension to qp-bosons

The above results admit direct extension to the case of the two-parameter deformed (or qp-) oscillators and thus to the qp-Bose gas model. For this, we use in analogy with (11) and (14) the Hamiltonian

$$H = \sum_i \omega_i N_i^{(qp)}. \tag{25}$$

With (25), the expression for general n-particle distribution functions is obtained (see the appendix for its derivation) as

$$\begin{gathered} \left\langle \left(A_i^\dagger\right)^n (A_i)^n \right\rangle = \frac{[\![n]\!]_{qp}!(\mathrm{e}^{\beta\omega_i} - 1)}{\prod_{r=o}^{n}(\mathrm{e}^{\beta\omega_i} - q^r p^{n-r})} \\ [\![m]\!]_{qp} \equiv \frac{q^m - p^m}{q - p} \qquad [\![m]\!]_{qp}! = [\![1]\!]_{qp}[\![2]\!]_{qp}\cdots[\![m-1]\!]_{qp}[\![m]\!]_{qp}. \end{gathered} \tag{26}$$

In the particular cases where $n = 1$ and $n = 2$ (note that $[\![2]\!]_{qp} = p + q$) this obviously yields the formulae

$$\begin{gathered} \left\langle A_i^\dagger A_i \right\rangle = \frac{(\mathrm{e}^{\beta\omega_i} - 1)}{(\mathrm{e}^{\beta\omega_i} - p)(\mathrm{e}^{\beta\omega_i} - q)} \\ \left\langle \left(A_i^\dagger\right)^2 (A_i)^2 \right\rangle = \frac{(p+q)(\mathrm{e}^{\beta\omega_i} - 1)}{(\mathrm{e}^{\beta\omega_i} - q^2)(\mathrm{e}^{\beta\omega_i} - pq)(\mathrm{e}^{\beta\omega_i} - p^2)} \end{gathered}$$

(note that the latter two formulae were also found in [25]).

From (26), after dividing it by $\left\langle A_i^\dagger A_i \right\rangle^n$, the most general result for the nth order qp-deformed extension of the intercept $\lambda^{(n)}$, omitting the i, follows as

$$\lambda_{q,p}^{(n)} \equiv \frac{\langle A^{\dagger n} A^n \rangle}{\langle A^\dagger A \rangle^n} - 1 = [\![n]\!]_{qp}! \frac{(\mathrm{e}^{\beta\omega} - p)^n (\mathrm{e}^{\beta\omega} - q)^n}{(\mathrm{e}^{\beta\omega} - 1)^{n-1} \prod_{k=0}^{n} (\mathrm{e}^{\beta\omega} - q^{n-k} p^k)} - 1 \tag{27}$$

which constitutes our main result. This provides a generalization not only to the case of nth order correlations but also to the two-parameter (qp-)deformation.

Let us give the asymptotical form of intercepts in this most general case, $\lambda_{q,p}^{(n)}$

$$\lambda_{q,p}^{(n),\mathrm{asympt}} = -1 + [\![n]\!]_{qp}! = -1 + \prod_{k=1}^{n} \left(\sum_{r=0}^{k} q^r p^{k-r} \right). \tag{28}$$

As observed, for each case of deformed bosons (the AC-type, the BM-type, and their qp-generalization) the asymptotics of the nth order intercept takes the form of the corresponding generalization of the usual n-factorial (the latter yields a pure Bose–Einstein n-particle correlation intercept).

Finally, let us specialize the obtained formulae to the case of q-bosons of BM type for $n = 3$, that is

$$\lambda_{\mathrm{BM}}^{(3)} = -1 + \frac{[2]_q [3]_q (\mathrm{e}^{2\beta\omega} - 2\,\mathrm{e}^{\beta\omega} \cos\theta + 1)^2}{(\mathrm{e}^{\beta\omega} - 1)^2 (\mathrm{e}^{2\beta\omega} - 2\,\mathrm{e}^{\beta\omega} \cos(3\theta) + 1)} \tag{29}$$

$$\lambda_{\mathrm{BM}}^{(3),\mathrm{asympt}} = -1 + [2]_q [3]_q = -1 + 2\cos\theta(2\cos\theta - 1)(2\cos\theta + 1). \tag{30}$$

In conclusion, we note that it would be of great interest and importance to make a detailed comparative analysis of the obtained results with the existing data for three-particle correlations of pions and kaons produced and registered in the experiments on relativistic heavy ion collisions, with the objective of drawing some implications concerning the immediate physical meaning and admissible values of the deformation parameters p, q. Details of such analysis will be presented elsewhere.

Appendix

Here we derive the general formula, see (26), for the (monomode) n-particle qp-bosonic distribution functions:

$$\langle a^{\dagger n} a^n \rangle = \frac{[\![n]\!]_{qp}!(\mathrm{e}^{\beta\omega} - 1)}{\prod_{r=0}^{n}(\mathrm{e}^{\beta\omega} - p^r q^{n-r})}. \tag{A.1}$$

For convenience, in (A.1) and below, we drop the mode-labelling subscript i and use $a^\dagger, a, N$ instead of $A^\dagger, A, N^{(qp)}$ respectively. The proof proceeds in a few steps. First let us derive the recursion relation

$$\langle a^{\dagger n} a^n \rangle = \langle a^{\dagger n-1} a^{n-1} p^N \rangle \frac{[\![n]\!]_{qp}}{(\mathrm{e}^{\beta\omega} - q^n) p^{n-1}}. \tag{A.2}$$

For this, we use qp-deformed commutation relations and evaluate the thermal averages:

$$\begin{aligned}
\langle a^{\dagger n} a^n \rangle &= \langle a^{\dagger n-1} a a^\dagger a^{n-1} \rangle \frac{1}{q} - \langle a^{\dagger n-1} p^N a^{n-1} \rangle \frac{1}{q} \\
&= \langle a^{\dagger n-1} a a^\dagger a^{n-1} \rangle \frac{1}{q} - \langle a^{\dagger n-1} a^{n-1} p^N \rangle \frac{1}{q p^{n-1}} \\
&= \langle a^{\dagger n-1} a^2 a^\dagger a^{n-2} \rangle \frac{1}{q^2} - \frac{1}{q}\left(\frac{1}{p^{n-1}} + \frac{1}{q p^{n-2}}\right) \langle a^{\dagger n-1} a^{n-1} p^N \rangle = \cdots \\
&= \langle a^{\dagger n-1} a^n a^\dagger \rangle \frac{1}{q^n} - \frac{1}{q}\left(\frac{1}{p^{n-1}} + \frac{1}{q p^{n-2}} + \cdots + \frac{1}{q^{n-1}}\right) \langle a^{\dagger n-1} a^{n-1} p^N \rangle \\
&= \langle a^{\dagger n-1} a^n a^\dagger \rangle \frac{1}{q^n} - \langle a^{\dagger n-1} a^{n-1} p^N \rangle \frac{[\![n]\!]_{qp}}{q^n p^{n-1}} \\
&= \langle a^{\dagger n} a^n \rangle \frac{\mathrm{e}^{\beta\omega}}{q^n} - \langle a^{\dagger n-1} a^{n-1} p^N \rangle \frac{[\![n]\!]_{qp}}{q^n p^{n-1}}.
\end{aligned} \tag{A.3}$$

From this equation (A.2) readily follows. After the kth iteration of this procedure we find

$$\langle a^{\dagger n-k} a^{n-k} p^{kN} \rangle = \langle a^{\dagger n-(k+1)} a^{n-(k+1)} p^{(k+1)N} \rangle \frac{[\![n-k]\!]_{qp}}{(\mathrm{e}^{\beta\omega} - q^{n-k} p^k) p^{n-(2k+1)}}. \tag{A.4}$$

Indeed,

$$\begin{aligned}
\langle a^{\dagger n-k} a^{n-k} p^{kN} \rangle &= \langle a^{\dagger n-(k+1)} a a^\dagger a^{n-k-1} p^{kN} \rangle \frac{1}{q} - \langle a^{\dagger n-(k+1)} p^N a^{n-(k+1)} p^{kN} \rangle \frac{1}{q} = \cdots \\
&= \langle a^{\dagger n-k} a^{n-k} p^{kN} \rangle \frac{\mathrm{e}^{\beta\omega}}{q^{n-k} p^k} \\
&\quad - \frac{1}{q}\left(\frac{1}{p^{n-1-k}} + \frac{1}{q p^{n-2-k}} + \cdots + \frac{1}{q^{n-k-1}}\right) \langle a^{\dagger n-(k+1)} a^{n-(k+1)} p^{(k+1)N} \rangle \\
&= \langle a^{\dagger n-k} a^{n-k} p^{kN} \rangle \frac{\mathrm{e}^{\beta\omega}}{q^{n-k} p^k} - \langle a^{\dagger n-(k+1)} a^{n-(k+1)} p^{(k+1)N} \rangle \frac{[\![n-k]\!]_{qp}}{q^{n-k} p^{n-(k+1)}}
\end{aligned} \tag{A.5}$$

that is equivalent to formula (A.4). Applying this formula step by step n times yields the relation

$$\langle a^{\dagger n} a^n \rangle = \frac{[\![n]\!]_{qp}!}{\prod_{r=0}^{n-1} (\mathrm{e}^{\beta\omega} - p^r q^{n-r}) \prod_{k=0}^{n-1} p^{n-(2k+1)}} \langle p^{nN} \rangle. \tag{A.6}$$

From the latter, with the account of

$$\langle p^{nN} \rangle = \frac{\mathrm{e}^{\beta\omega} - 1}{\mathrm{e}^{\beta\omega} - p^n} \qquad \prod_{k=0}^{n-1} p^{n-(2k+1)} = 1 \tag{A.7}$$

we finally arrive at the desired formula (A.1), for the higher order (n-particle) monomode distribution functions of the model of qp-Bose gas.

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